

Matter evolution in Burgulence

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Abstract

In inviscid solutions of the forced Burgers equation the matter accumulates in the shock discontinuities. We describe the limit motion of particles everywhere including the shocks as the trajectories of a discontinuous velocity field being a generalization of the gradient of the limit potential. The latter is not differentiable but satisfies some convexity properties which guarantee the existence of the gradient. It turns out that for such discontinuous gradient ordinary differential equations there are natural existence, uniqueness, and continuity theorems. These general results are applied for investigation of formation and motion in plane of massive points which are interpreted as various clusters in the adhesion model of the Universe.

Keywords: Burgers equation, shocks, massive points, clusters, singularities, transitions.

1 Introduction

The subject of this paper is the matter evolution in limit potential solutions of the Burgers equation with vanishing viscosity and external potential force. The Burgers equation is just the Navier–Stokes equation without the pressure term – its theory is well described in the survey [1].

It is well known that in such inviscid potential solutions there can be shocks, i.e. velocity discontinuities. They appear even if the initial condition and the external force are *smooth* – here and further this term means infinite differentiability. Generically in this smooth case shocks are smooth hypersurfaces with prescribed singularities. “Generically” means that other singularities can be killed by arbitrarily small perturbation of the smooth initial condition.

In plane such a generic shock is a smooth curve with triple nodes and end points looking like shown in Figure 1. It can experience the transitions which are shown in Figure 2.

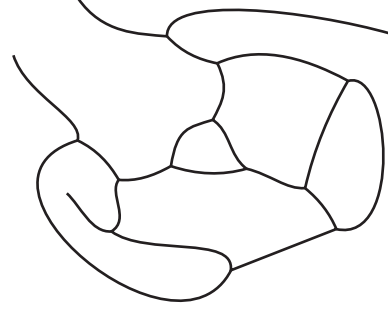


Figure 1. Example of shock in plane

Limit potential solutions of the Burgers equation describe motion when the particles cannot pass through each other and adhere on shocks. It happens because a particle cannot leave the shock. More precisely, a particle trajectory ending outside of the shock lies outside of the shock as well, but a trajectory beginning outside of the shock can end on the shock. In other words, the matter accumulates in the shocks where the density is infinite. This is the so-called adhesion model of matter evolution in the Universe describing the formation of cellular structure of the matter (see, for example, [2]) – the adhesion of particles is a result of interaction between them described by the vanishing viscosity.

But what is the motion of particles on shocks? This question is answered by the present paper. Its main results are the following.

1) When the viscosity is positive the trajectory of any particle is well defined because the velocity field is smooth. It turns out there exists a limit of the trajectory as the viscosity vanishes. Such limit trajectories describe the motion of particles in the inviscid solu-

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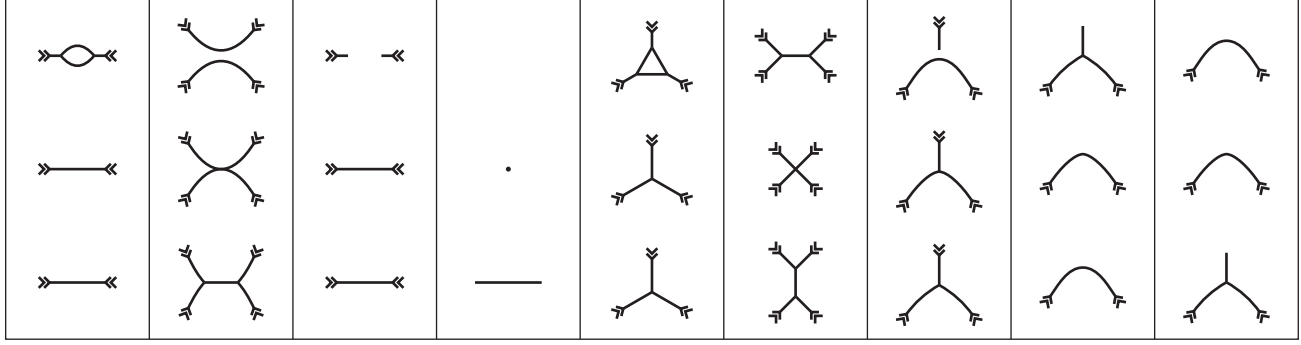


Figure 2. Generic transitions of shocks in plane

tion. The following uniqueness theorem is true: there is only one limit trajectory beginning at a given point but there can be a few limit trajectories ending at it. Of course, a few trajectories can end only at a point of the shock.

2) How to find the limit trajectory of a given particle? It turns out that the limit trajectory is a solution of the Cauchy problem for a velocity field defined by the limit potential. The velocity field is discontinuous but, nevertheless, the Cauchy problem has a unique solution.

Besides, the above general results are applied for investigation of formation and motion in plane of massive points (points with positive mass) which are interpreted as various *clusters* in the adhesion model of the Universe. It is natural to assume that on the shock outside of its singularities the matter is distributed with positive linear density and the nodes are massive points. However, this idea is not correct!

Indeed, the velocity field of the limit trajectories is smooth on the shock outside of its singularities and a cluster cannot appear there. But a cluster cannot appear at a node with an acute angle too because the field of the relative (with respect to the node) velocities looks like shown in Figure 3 on the left. So particles pass through such a node and the matter does not accumulate here. Otherwise, if all angles of a node are obtuse then the matter is trapped at the node and it is a growing cluster as it is shown in Figure 3 in the middle. Its right side shows that a cluster cannot appear at an end point of the shock as well.

So, a cluster is born when an acute node turns into an obtuse one – see the left transition in Figure 4. After the opposite transformation the cluster stops growing and leaves the node – this is the left transition in Figure 5. (In our figures such stable clusters are shown by white disks, growing clusters – by black disks.) A stable cluster travels along the shock and, in particular,

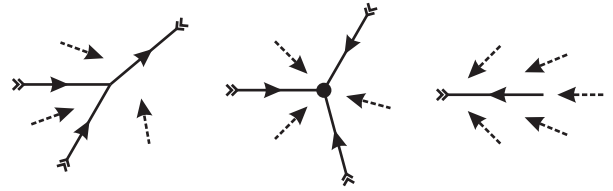


Figure 3. Velocities of particles around acute node, growing cluster at obtuse node, and end point

can pass through an acute node and be absorbed by a growing cluster (the left transitions in Figures 6 and 7 respectively).

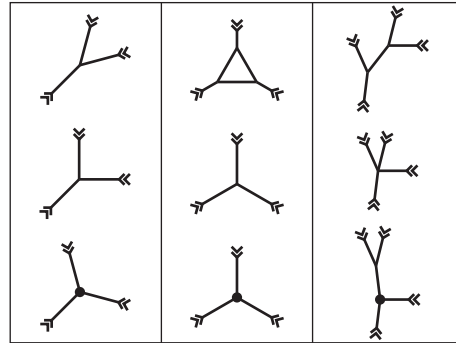


Figure 4. Cluster is born and starts growing

The transitions in the middle and on the right from Figures 4–7 show what happens with clusters when transitions of shocks occur. Clusters can be involved only in the fifth and sixth transitions from Figure 2 because all nodes of the other ones are acute. Besides, generically a stable cluster cannot come to a transition but can appear after it as shown in Figure 5 and 7.

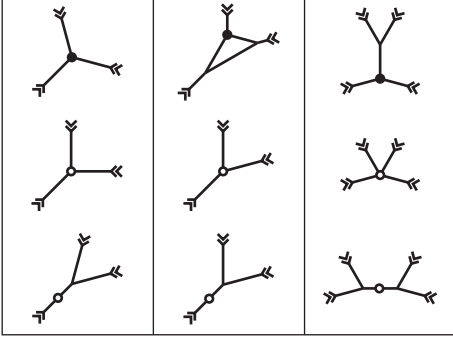


Figure 5. Cluster stops growing and leaves node

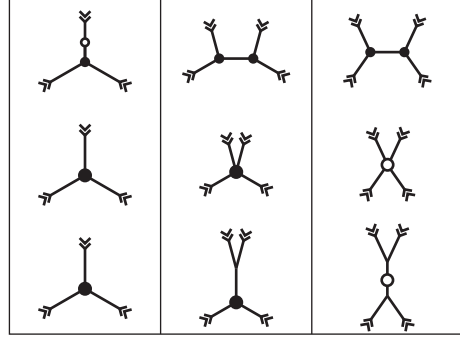


Figure 7. Two clusters adhere

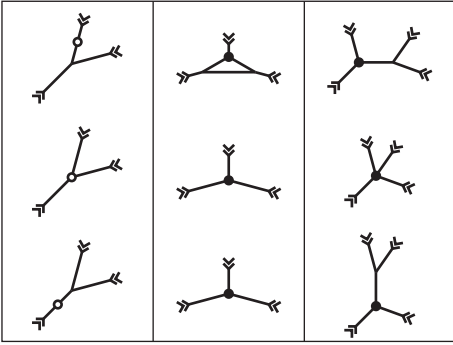


Figure 6. Cluster travels through nodes

I am very grateful to U. Frisch for calling my attention to the problem as well as to him, J. Bec, M. Blank, K. Khanin, R. Mohayaee, and A. Sobolevsky for fruitful discussions.

2 Existence of limit trajectories

So, we consider a material d -dimensional medium whose velocity is potential and described by the Burgers equation with the potential force term:

$$\begin{cases} \mathbf{v}_t^\nu + (\mathbf{v}^\nu \cdot \nabla) \mathbf{v}^\nu = -\nabla U + \nu \Delta \mathbf{v}^\nu \\ \mathbf{v}^\nu = \nabla \psi^\nu \\ \psi^\nu(\mathbf{x}, 0) = \varphi_0(\mathbf{x}) \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^d$ is a point of the medium, $\mathbf{v}^\nu(\mathbf{x}, t)$ is the velocity at the point \mathbf{x} at the time t , $\nu > 0$ is the viscosity of the medium, $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$ is the usual ∇ -operator in \mathbb{R}^d , and $\Delta = \nabla \cdot \nabla$ is the Laplacian. The potential ψ^ν of the velocity field \mathbf{v}^ν is defined with respect to a function of time which can be chosen so that the following equation is satisfied:

$$\psi_t^\nu + \frac{1}{2} \nabla \psi^\nu \cdot \nabla \psi^\nu + U = \nu \Delta \psi^\nu. \quad (1)$$

The force potential U and the initial condition φ_0 are assumed to be *smooth*. (Everywhere in the present paper it means infinite differentiability.) Let φ be the *limit solution* as the viscosity vanishes.

$$\varphi(\mathbf{x}, t) = \lim_{\nu \rightarrow 0} \psi^\nu(\mathbf{x}, t). \quad (2)$$

According to Theory of PDE, the potential ψ^ν is smooth if $\nu > 0$ and $t \geq 0$. As it has been mentioned before, the limit potential is continuous, but its gradient field can have discontinuities (shocks).

We consider the periodical case. It means that all data – the given force potential U , the initial condition φ_0 , the velocity field \mathbf{v}^ν , its potential ψ^ν , and the limit potential – are assumed to be space-periodical. In other words, we consider our equations on torus.

REMARK. The periodicity requirement is technical. Informally speaking, it is needed to guarantee that nothing goes to the infinity and nothing comes from the infinity for a finite period. Otherwise, let this informal requirement be satisfied and we want to apply our results in a finite domain of the space-time. Then we can always consider our data as periodical with an enough large period. Indeed, if we change them enough far we will not be able to observe the influence in our finite domain.

Let $y^\nu : [0, +\infty) \rightarrow \mathbb{R}^d$ be the trajectory beginning at an initial point $\mathbf{a} \in \mathbb{R}^d$, that is the solution of the following Cauchy problem:

$$\dot{y}^\nu(t) = \nabla \psi^\nu(y^\nu(t), t), \quad y^\nu(0) = \mathbf{a}$$

which has a unique solution because the right side of the ordinary differential equation is a smooth space-periodical vector field.

THEOREM 1. Existence: *For any initial point \mathbf{a} there exists a limit trajectory $x : [0, +\infty) \rightarrow \mathbb{R}^d$*

$$x(t) = \lim_{\nu \rightarrow 0} y^\nu(t), \quad x(0) = y^\nu(0) = \mathbf{a}.$$

The convergence is uniform on any segment $[0, T]$ and the limit trajectory is continuous.

Uniqueness: If two limit trajectories pass through the same point at some time then they coincide **after** that time:

$$x_1(t_*) = x_2(t_*) \Rightarrow x_1(t) = x_2(t) \quad \forall t \geq t_*.$$

(But they may not coincide **before** the time: $x_1(t) \neq x_2(t)$ for $t < t_*$.)

Continuity: The point $x(t)$ is a continuous function of the time t and the initial point \mathbf{a} .

COROLLARY. For any point \mathbf{x}_* and time $t_* \geq 0$ there is a limit trajectory $x : [0, +\infty) \rightarrow \mathbb{R}^d$ passing through the point at the time: $x(t_*) = \mathbf{x}_*$.

PROOF. According to Theory of ODE, for any ν there exists a trajectory y^ν such that $y^\nu(t_*) = \mathbf{x}_*$ (because \mathbf{v}^ν is smooth). Taking into account that our torus is compact we can choose a sequence $\nu_n \rightarrow 0$ as $n \rightarrow \infty$ such that the sequence $y^{\nu_n}(0)$ converges to a point \mathbf{a}_* as $n \rightarrow \infty$. For the limit trajectory x with the initial point $x(0) = \mathbf{a}_*$ we get $x(t_*) = \mathbf{x}_*$. \square

3 Differential equation for limit trajectories

THEOREM 2. The derivative

$$\varphi'_{\mathbf{x},t}(\mathbf{q}, 0) = \lim_{\lambda \rightarrow +0} \frac{\varphi(\mathbf{x} + \lambda \mathbf{q}, t) - \varphi(\mathbf{x}, t)}{\lambda}$$

of the limit potential along any space direction \mathbf{q} exists and can be presented as the minimum of linear functions:

$$\varphi'_{\mathbf{x},t}(\mathbf{q}, 0) = \min_{\mathbf{p} \in k_{\mathbf{x},t}} \mathbf{p} \cdot \mathbf{q} \quad (3)$$

where $k_{\mathbf{x},t}$ is a compact set of momenta which depends on the point (\mathbf{x}, t) of the space-time.

Besides, any limit trajectory from Theorem 1 satisfies the differential equation

$$x^+(t) = \mathbf{u}(x(t), t)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the center of the minimal ball containing the set $k_{\mathbf{x},t}$ and the left side is the one-way derivative

$$x^+(t) = \lim_{\lambda \rightarrow +0} \frac{x(t + \lambda) - x(t)}{\lambda}.$$

REMARK. The set $k_{\mathbf{x},t}$ consists of the limit velocities at the time t at points which are outside of the shock and tend to the point \mathbf{x} .

Theorem 2 is proved in Section 7 and can be briefly explained in the following way. When $\nu \neq 0$ the velocity $\mathbf{v}^\nu(\mathbf{x}, t) = \nabla \psi^\nu(\mathbf{x}, t)$ is the solution of the following minimum problem:

$$|\mathbf{q}|^2/2 - \psi^{\nu'}_{\mathbf{x},t}(\mathbf{q}, 1) \rightarrow \min_{\mathbf{q}}.$$

It turns out that this minimum principle remains valid for the limit potential. Namely, the limit velocity $\mathbf{u}(\mathbf{x}, t)$ is the solution of the same minimum problem for the limit potential:

$$|\mathbf{q}|^2/2 - \varphi'_{\mathbf{x},t}(\mathbf{q}, 1) \rightarrow \min_{\mathbf{q}}. \quad (4)$$

Proving this principle in Section 6 we do not use that the potentials ψ^ν are solutions of the Burgers equation – it is only important that their second derivatives are uniformly bounded above.

REMARK. The principle (4) is not variational because there is no an integral functional to be minimized by this principle. In fact, it just generalizes the notion of a gradient for a some class of non-smooth functions of \mathbf{x} and t (see Section 6 for details). But such the generalized gradient depends on the behavior of the function at closed times after t . It does not happen if the function is smooth – then we get the usual gradient defined completely by the first space derivatives of the function.

Let us show how the principle (4) implies Theorem 2. It is well known that in the case when the initial condition φ_0 is smooth there is the so-called minimum representation

$$\varphi(\mathbf{x}, t) = \min_{\xi} \{F(\xi, \mathbf{x}, t)\}$$

where F is a family of smooth solutions of the Hamilton–Jacobi equation:

$$F_t(\xi, \mathbf{x}, t) + |\nabla_{\mathbf{x}} F(\xi, \mathbf{x}, t)|^2/2 + U(\mathbf{x}, t) = 0$$

depending smoothly on a parameter ξ . (In other words, F is a smooth function of its variables ξ , \mathbf{x} , and t .) It immediately implies that the derivative of the limit potential along a direction of the space-time

$$\varphi'_{\mathbf{x},t}(\mathbf{q}, \tau) = \lim_{\lambda \rightarrow +0} \frac{\varphi(\mathbf{x} + \lambda \mathbf{q}, t + \lambda \tau) - \varphi(\mathbf{x}, t)}{\lambda}$$

can be presented as the minimum of some solutions of the Hamilton–Jacobi equation freezed at the point (\mathbf{x}, t) :

$$\varphi'_{\mathbf{x},t}(\mathbf{q}, \tau) = \min_{\mathbf{p} \in K_{\mathbf{x},t}} \{\mathbf{p} \cdot \mathbf{q} - \tau |\mathbf{p}|^2/2 - U(\mathbf{x}, t) \tau\} \quad (5)$$

where $K_{\mathbf{x},t}$ is a compact set of momenta which depends on the point (\mathbf{x}, t) of the space-time. Substituting here $\tau = 0$ and comparing with the formula (3) we get that the sets $k_{\mathbf{x},t}$ and $K_{\mathbf{x},t}$ has the same convex hull.

Applying the principle (4) we get that the value

$$\begin{aligned} & |\mathbf{q}|^2/2 - \varphi'_{\mathbf{x},t}(\mathbf{q}, 1) = \\ & = |\mathbf{q}|^2/2 - \min_{\mathbf{p} \in K_{\mathbf{x},t}} \{ \mathbf{p} \cdot \mathbf{q} - |\mathbf{p}|^2/2 - U(\mathbf{x}, t) \} = \\ & = \max_{\mathbf{p} \in K_{\mathbf{x},t}} \{ |\mathbf{p} - \mathbf{q}|^2/2 \} - U(\mathbf{x}, t) \end{aligned}$$

attains its minimum at the center of the minimal ball containing the set $k_{\mathbf{x},t}$ because the latter has the same convex hull as the set $K_{\mathbf{x},t}$.

REMARK. The equivalence of Theorem 2 and the principle (4) for the limit solutions of the Burgers equation has been observed independently on the author of the present paper by K. Khanin and A. Sobolevsky.

4 Proof of Theorem 1

LEMMA 1. *For any $T > 0$ there exists a constant $C(T)$ bounding for all $\nu > 0$ and $0 \leq t \leq T$ the second derivative of the potential in any direction of the space-time*

$$\psi''_{QQ} \leq C(T),$$

where $Q = (\mathbf{q}, \tau)$, $\mathbf{q} = (q_1, \dots, q_d)$, $|\mathbf{q}|^2 + \tau^2 = 1$.

PROOF. This is the standard maximum principle for the second derivative ψ''_{QQ} that satisfies the equation

$$\psi''_{QQt} + \nabla \psi''_Q \cdot \nabla \psi''_Q + \nabla \psi'' \cdot \nabla \psi''_{QQ} + U_{QQ} = \nu \Delta \psi''_{QQ}$$

which is a consequence of (1) and implies the inequality

$$\psi''_{QQt} + \nabla \psi'' \cdot \nabla \psi''_{QQ} + B \leq \nu \Delta \psi''_{QQ}$$

where

$$B(T) = \min U_{QQ}(\mathbf{x}, t), \quad |Q| = 1, \quad \mathbf{x} \in \mathbb{R}^d, \quad 0 \leq t \leq T.$$

Then the following inequality holds:

$$\eta_t + \nabla \psi'' \cdot \nabla \eta \leq \nu \Delta \eta, \quad \eta = \psi''_{QQ} + Bt.$$

So, when the function η attains its maximal value for $t \in [0, T]$ we get the inequality $\eta_t \leq 0$ which shows that the maximal value can be attained only if $t = 0$. Therefore,

$$\eta - Bt \leq \max \{ \eta|_{t=0} \} - Bt$$

that means

$$\psi''_{QQ} \leq \max \{ \psi''_{QQ}|_{t=0} \} - Bt$$

but $\max \{ \psi''_{QQ}|_{t=0} \}$ is defined by the initial condition φ_0 and the force potential U because

$$\psi''_{QQ} = \psi''_{\mathbf{q}\mathbf{q}} + 2\tau \psi''_{\mathbf{q}t} + \tau^2 \psi''_{tt}$$

where according to (1):

$$\psi''_{\mathbf{q}t} = \nu \Delta \psi''_{\mathbf{q}} - \nabla \psi'' \cdot \nabla \psi''_{\mathbf{q}} - U_{\mathbf{q}},$$

$$\psi''_{tt} = \nu \Delta \psi''_t - \nabla \psi'' \cdot \nabla \psi''_t - U_t,$$

$$\psi''_t = \nu \Delta \psi'' - \frac{1}{2} \nabla \psi'' \cdot \nabla \psi'' - U.$$

If, for example,

$$C(T) = \max \{ \varphi_{QQ} \} + |B(T)|T,$$

we get the statement. \square

LEMMA 2. *For any $T > 0$ the convergence (2) is uniform on $\mathbb{R}^d \times [0, T]$.*

PROOF. According to Lemma 1 the functions

$$C(T) (|\mathbf{x}|^2 + t^2)/2 - \psi''(\mathbf{x}, t)$$

are convex on $\mathbb{R}^d \times [0, T]$. Hence, their convergence as $\nu \rightarrow 0$ is uniform on any compact subset according to Theorem 4 from Section 5. But the functions $\psi''(\mathbf{x}, t)$ are space-periodical and their convergence is uniform on $\mathbb{R}^d \times [0, T]$. \square

We are proving Theorem 1. Let $\alpha, \varepsilon > 0$. Lemma 2 implies that for any sufficiently small ν and ν_*

$$|\psi(\mathbf{x}) - \psi_*(\mathbf{x})| < \varepsilon$$

where $\psi(\mathbf{x}) = \psi''(\mathbf{x}, t)$, $\psi_*(\mathbf{x}) = \psi''_*(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^d$, and $t \in [0, T]$. We want to get a uniform upper bound for the square of the distance between the corresponding trajectories:

$$R(t) = |y(t) - y_*(t)|^2$$

where $y(t) = y''(t)$, $y_*(t) = y''_*(t)$,

$$y(0) = \mathbf{a}, \quad y_*(0) = \mathbf{a}_*, \quad |\mathbf{a} - \mathbf{a}_*| < \alpha.$$

According to Lemma 1:

$$\psi_*(y) - \psi_*(y_*) \leq \nabla \psi_*(y_*) \cdot (y - y_*) + C |y - y_*|^2/2,$$

$$\psi(y_*) - \psi(y) \leq \nabla \psi(y) \cdot (y_* - y) + C |y_* - y|^2/2.$$

Adding the inequalities we get:

$$-2\varepsilon < -(\nabla \psi(y) - \nabla \psi_*(y_*)) \cdot (y - y_*) + C |y - y_*|^2,$$

or

$$(\nabla \psi(y) - \nabla \psi_*(y_*)) \cdot (y - y_*) < 2\varepsilon + C |y - y_*|^2, \quad (6)$$

that gives

$$\dot{R}(t) < 4\varepsilon + 2C R(t).$$

Solving the differential inequality and taking into account that $R(0) < \alpha$, we get:

$$R(t) < 2\varepsilon \frac{e^{2Ct} - 1}{C} + \alpha e^{2Ct},$$

or in the special case $C = 0$:

$$R(t) < 4\varepsilon t + \alpha.$$

The inequalities give the required uniform upper bound.

5 Convex functions

Let M be a convex subset of an m -dimensional affine space. (It means that for any two points of M the segment connecting them belongs to M as well.) A function $f : M \rightarrow \mathbb{R}$ is called *convex* if it satisfies the inequality

$$f(\alpha X + \beta Y) \leq \alpha f(X) + \beta f(Y)$$

for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and any points $X, Y \in M$. A smooth function is convex if and only if its second derivative along any direction is non-negative.

It is well known – see, for example [3] – that convex functions have many good properties:

THEOREM 3. *Let $M \in \mathbb{R}^m$ be an open convex subset and $f : M \rightarrow \mathbb{R}$ be a convex function.*

Continuity: *f is continuous on M .*

Differentiability: *f has a finite derivative along any direction Q at any point $X \in M$:*

$$f'_X(Q) = \lim_{\lambda \rightarrow +0} \frac{f(X + \lambda Q) - f(X)}{\lambda}.$$

Moreover,

$$f(X + Q) = f(X) + f'_X(Q) + o(|Q|), \quad Q \rightarrow 0$$

where the derivative $f'_X(Q)$ is a convex homogeneous function of Q :

$$f'_X(\lambda Q) = \lambda f'_X(Q), \quad \lambda \geq 0$$

and can be presented as

$$f'_X(Q) = \max_{P \in \mathcal{D}_X(f)} P \cdot Q$$

where the set $\mathcal{D}_X(f)$ is convex and consists of the sub-differentials of the function f at the point X .

Sub-differential boundedness: *The sub-differentials $\mathcal{D}_X(f)$ is uniformly bounded if X belongs to a compact subset of M .*

This theorem is proved in [3] (see theorems 10.1 and 23.1).

THEOREM 4. *Let $M \in \mathbb{R}^m$ be an open convex subset and $f^\nu : M \rightarrow \mathbb{R}$ be a family of convex functions depending on a parameter ν .*

Uniform convergence: *If the family converges*

$$f_*(X) = \lim_{\nu \rightarrow 0} f^\nu(X)$$

then the limit function f_ is convex on M and the convergence is uniform on any compact subset of M .*

Uniform derivative boundedness: *If the family f^ν is uniformly bounded on M then the family of the sub-differentials $\mathcal{D}_X(f^\nu)$ is uniformly bounded if X belongs to a compact subset of M .*

6 Gradient differential equations

Let $M \subset \mathbb{R}^m$ be an open convex subset; a potential $\varphi : M \rightarrow \mathbb{R}$ be the difference of a semi-definite quadratic form e and a convex function f :

$$\varphi(X) = e(X) - f(X), \quad X \in M;$$

$\mathcal{T} = \mathbb{R}^m$ be the tangent space to M or the space of velocities; $\mathcal{T}^* = \mathbb{R}^{m*}$ be the cotangent space to M or the space of momenta; and a Hamiltonian $h : \mathcal{T}^* \rightarrow \mathbb{R}$ be a smooth convex function of momenta.

The derivative of φ can be written in the following form:

$$\varphi'_X(Q) = \min_{P \in \mathcal{D}_X} P \cdot Q, \quad P \in \mathcal{T}^*, \quad Q \in \mathcal{T}$$

where $\mathcal{D}_X = \mathcal{D}_X(\varphi) \subset \mathcal{T}^*$ is a compact convex set of momenta – see Theorem 3 for details. The momenta from the set \mathcal{D}_X are called *sub-differentials* of the potential φ at the point X .

Let the convex Hamiltonian h attain its minimal value on \mathcal{D}_X at a point $P_X \in \mathcal{D}_X$ of the convex set of the sub-differentials of the potential φ .

DEFINITION. 1) *Hamiltonian form:* The velocity $Q_X = h'_{P_X} \in \mathcal{T}$ is denoted by $\nabla_h \varphi(X)$ and called the *h -gradient* of the potential φ at the point X . Here h'_{P_X} is the differential of the Hamiltonian h at the point P_X .

2) *Lagrangian form:* The h -gradient Q_X is the minimum point of the function

$$l(Q) - \varphi'_X(Q), \quad l(Q) = \max_P \{P \cdot Q - h(P)\}$$

where l is the Lagrangian being the Legendre transformation of the Hamiltonian h .

In order to show that the Hamiltonian and Lagrangian forms are equivalent, let us note that

$$\varphi'_X(Q_X) = P_X \cdot Q_X \quad (7)$$

where $Q_X = \nabla_h \varphi(X)$. Indeed,

$$\min_{P \in \mathcal{D}_X} P \cdot Q_X = P_X \cdot Q_X$$

because P_X is a minimum point of the smooth function h on the convex set \mathcal{D}_X and for any $P \in \mathcal{D}_X$ we get $(P - P_X) \cdot h'_{P_X} \geq 0$. Besides,

$$h(P) \geq P \cdot Q - l(Q)$$

because $h(P) = \max_Q \{P \cdot Q - l(Q)\}$. After minimizing we get

$$h(P_X) \geq \min_{P \in \mathcal{D}_X} P \cdot Q - l(Q) = \varphi'_X(Q) - l(Q).$$

But $h(P_X) = P_X \cdot Q_X - l(Q_X)$ because $Q_X = h'_{P_X}$ that gives

$$h(P_X) = \varphi'_X(Q_X) - l(Q_X)$$

together with (7). Comparing the last two displayed formulas we get

$$\varphi'_X(Q_X) - l(Q_X) \geq \varphi'_X(Q) - l(Q)$$

that proves the equivalence of our Hamiltonian and Lagrangian forms of defining the h -gradient.

REMARK. If the point P_X is not defined uniquely the h -gradient does not depend on it. Indeed, if a smooth convex function has the same value on a segment, its differentials coincide at the points of the segment.

THEOREM 5. *Let the Hamiltonian be the sum of a linear form and a positive semi-definite quadratic form:*

$$h(P) = h_1(P) + h_2(P);$$

e be a positive semi-definite quadratic form on \mathbb{R}^m and $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a potential with bounded sub-differentials such that the difference $e - \varphi$ is a convex function on \mathbb{R}^m .

Then the Cauchy problem

$$\mathcal{X}^+(t) = \nabla_h(\varphi)(\mathcal{X}(t)), \quad \mathcal{X}(0) = X_0,$$

where the left side of the differential equation is the one-way derivative

$$\mathcal{X}^+(t) = \lim_{\lambda \rightarrow +0} \frac{\mathcal{X}(t + \lambda) - \mathcal{X}(t)}{\lambda},$$

has a unique global solution which depends continuously on the initial point X_0 and the potential φ provided that the quadratic form e is fixed.

More precisely, it means that there exists a unique trajectory $\mathcal{X} : [0, +\infty) \rightarrow \mathbb{R}^m$ satisfying the differential equation for any $t \geq 0$ and the initial point $\mathcal{X}(0) = X_0$. Moreover, if the quadratic form e is fixed then the point $\mathcal{X}(t)$ depends continuously on the time t , the initial point X_0 , and the potential φ with respect to the compact-open topology.

REMARK. Of course, Theorem 5 has local variants but it is more convenient to formulate and prove it globally.

REMARK. Theorem 5 looks correct for any smooth convex Hamiltonian h but the author has failed to find its proof in this case.

7 Proof of Theorem 2

Theorem 2 follows from Theorem 5 applied in the strip $\mathbb{R}^d \times [0, T]$ of the affine space-time to the potentials ψ^ν and φ . Let

$$X = (\mathbf{x}, t), \quad P = (\mathbf{p}, \sigma), \quad Q = (\mathbf{q}, \tau),$$

$$h(\mathbf{p}, \sigma) = |\mathbf{p}|^2/2 + \sigma,$$

$$e(\mathbf{x}, t) = C(T) (|\mathbf{x}|^2 + t^2)/2$$

where $C(T)$ is the constant from Lemma 1. According to Lemma 1, the potentials ψ^ν satisfy the conditions of Theorem 5 in the strip. Besides,

$$\nabla_h \psi^\nu = (\nabla \psi^\nu, 1)$$

where $\nabla \psi^\nu$ is the usual gradient of the smooth potential ψ^ν . Theorem 4 shows that the limit potential φ satisfies the conditions of Theorem 5 as well and the convergence $\psi^\nu \rightarrow \varphi$ as $\nu \rightarrow +0$ is uniform in the strip. So, the derivative $\varphi'_{\mathbf{x},t}$ exists and it remains to show that

$$\nabla_h \varphi(\mathbf{x}, t) = (\mathbf{u}(\mathbf{x}, t), 1) \quad (8)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the center of the minimal ball containing the set $k_{\mathbf{x},t}$.

REMARK. Formally speaking, we cannot apply Theorem 5 in the strip but we can always extend our potentials up to functions on the space-time satisfying the conditions of Theorem 5.

In order to show the equality (8) we can use the both forms of the definition h -gradient from Section 6.

Hamiltonian form: According to (5), the set $\mathcal{D}_{\mathbf{x},t}$ of the sub-differentials of the limit potential φ at the point (\mathbf{x}, t) is the convex hull of the set

$$\{(\mathbf{p}, \sigma) \mid \mathbf{p} \in K_{\mathbf{x},t}, |\mathbf{p}|^2/2 + \sigma + U(\mathbf{x}, t) = 0\}.$$

Hence, the Hamiltonian $h(\mathbf{p}, \sigma) = |\mathbf{p}|^2/2 + \sigma$ attains its minimum on $\mathcal{D}_{\mathbf{x},t}$ at some point $(\mathbf{u}(\mathbf{x},t), \sigma_*)$ where $\mathbf{u}(\mathbf{x},t)$ is the center of the minimal ball containing the set $K_{\mathbf{x},t}$. But substituting into (5) $\tau = 0$ and comparing with the formula (3) we get that the sets $k_{\mathbf{x},t}$ and $K_{\mathbf{x},t}$ has the same convex hull. Therefore, $\mathbf{u}(\mathbf{x},t)$ is the center of the minimal ball containing the set $k_{\mathbf{x},t}$ as well, and we get

$$\nabla_h \varphi(\mathbf{x}, t) = h'_{\mathbf{u}(\mathbf{x},t), \sigma_*} = (\mathbf{u}(\mathbf{x},t), 1).$$

Lagrangian form: The Lagrangian

$$l(\mathbf{q}, \tau) = \begin{cases} |\mathbf{q}|^2/2 & \text{if } \tau = 1 \\ +\infty & \text{if } \tau \neq 1 \end{cases}$$

is the Legendre transformation of the Hamiltonian $h(\mathbf{p}, \sigma) = |\mathbf{p}|^2/2 + \sigma$. This means that the Lagrangian form of the definition of the h -gradient from Section 6

$$l(\mathbf{q}, \tau) - \varphi'_{\mathbf{x},t}(\mathbf{q}, \tau) \rightarrow \min_{\mathbf{q}, \tau}$$

is nothing but the principle (4).

8 Proof of Theorem 5

According to Theory of ODE, if the potential φ is smooth then the Cauchy problem has a unique solution and the boundedness of the differentials of the potential φ guarantees that it is defined globally.

Moreover, our Cauchy problem has a solution if the potential φ is homogeneous of the degree 1 and *concave*. (The last word means that the function $-\varphi$ is convex.) This key observation is formulated in the following lemma.

LEMMA 3. *If φ is a concave homogeneous function:*

$$\varphi(\lambda X) = \lambda \varphi(X) \quad \forall \lambda \geq 0,$$

then the trajectory $\mathcal{X} : [0, +\infty) \rightarrow \mathbb{R}^m$, $\mathcal{X}(t) = t \nabla_h \varphi(0)$ is a solution of the Cauchy problem

$$\mathcal{X}^+(t) = \nabla_h \varphi(\mathcal{X}(t)), \quad \mathcal{X}(0) = 0.$$

This lemma is true for any smooth convex Hamiltonian; in fact, the following proof uses its smoothness only, but the convexity is needed for the uniqueness of the h -gradient.

PROOF. Firstly, $\mathcal{D}_X \subset \mathcal{D}_0$ for any point $X \in \mathbb{R}^m$ because our potential φ is concave and homogeneous of the degree 1. (Any sub-differential at any point is a sub-differential at 0.)

Secondly, if the smooth Hamiltonian h attains its minimal value on \mathcal{D}_0 at a point $P_0 \in \mathcal{D}_0$ then $P_0 \in$

$\mathcal{D}_{\mathcal{X}(t)}$ for any $t \geq 0$. It immediately follows from the equality

$$\varphi(\mathcal{X}(t)) = P_0 \cdot \mathcal{X}(t)$$

because $\varphi(X) \leq P_0 \cdot X$. The last equality follows from (7). Independently on this reference: $\mathcal{X}(t) = th'_{P_0}$ and for $t \geq 0$

$$\varphi(th'_{P_0}) = \min_{P \in \mathcal{D}_0} P \cdot th'_{P_0} = P_0 \cdot th'_{P_0}.$$

Indeed, for any $P \in \mathcal{D}_0$ we get $(P - P_0) \cdot h'_{P_0} \geq 0$ because P_0 is a minimum point of the smooth Hamiltonian h on the convex set \mathcal{D}_0 .

Therefore, the inclusions $P_0 \in \mathcal{D}_{\mathcal{X}(t)} \subset \mathcal{D}_0$ show that P_0 is a minimum point of the Hamiltonian h on the set $\mathcal{D}_{\mathcal{X}(t)}$ which implies that $\nabla_h \varphi(\mathcal{X}(t)) = \nabla_h \varphi(0)$. \square

Let Φ_e be the space of all potentials φ with bounded sub-differentials such that the differences $e - \varphi$ are convex functions (i. e., Φ_e consists of the potentials satisfying the conditions of the theorem) and

$$\mathbf{g} : \Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad (\varphi, t, X_0) \mapsto \mathbf{g}_\varphi^t(X_0)$$

be the mapping sending a potential φ , a time $t \geq 0$, and an initial point X_0 to the value of the solution of our Cauchy problem at the time t . In other words, the trajectory

$$\mathcal{X}(t) = \mathbf{g}_\varphi^t(X_0)$$

is a solution of the Cauchy problem.

A priori, the mapping \mathbf{g} can be many-valued and defined not everywhere. Let $D(\mathbf{g}) \subset \Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m$ is the domain of definition of the mapping \mathbf{g} .

LEMMA 4. *If the Hamiltonian is the sum of a linear form and a positive semi-definite quadratic form:*

$$h(P) = h_1(P) + h_2(P)$$

then the mapping \mathbf{g} is one-valued and continuous on $D(\mathbf{g})$ with respect to the compact-open topology in the space Φ_e .

REMARK. In the case $h(P) = |P|^2/2$ the proof of Theorem 1 proves, in fact, Lemma 4 as well. The key place is the inequality (6).

PROOF. We are showing the continuity of the mapping \mathbf{g} at a point $(\varphi_*, t_*, X_*) \in \Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m$. Let us consider an open bounded set K consisting the trajectory

$$\mathcal{X}_*(t) = \mathbf{g}_{\varphi_*}^t(X_*), \quad t \in [0, t_*]$$

– it can always be done because the h -gradient of the potential $\varphi_* \in \Phi_e$ is bounded. Let an open bounded

convex set M contain the closure \bar{K} of the set K and φ be any potential such that

$$|\varphi(X) - \varphi_*(X)| < \varepsilon \quad \forall X \in \bar{M}. \quad (9)$$

According to Theorem 4, the last condition guarantees that the sub-differentials of all such potentials at all points of \bar{K} are bounded by a constant B .

Let us consider another trajectory

$$\mathcal{X}(t) = \mathbf{g}_\varphi^t(X), \quad X \in K$$

and introduce the following notation:

$$\delta\mathcal{X}(t) = \mathcal{X}(t) - \mathcal{X}_*(t), \quad \delta\mathcal{P}(t) = \mathcal{P}(t) - \mathcal{P}_*(t),$$

where

$$\mathcal{P}(t) = P_{\mathcal{X}(t)}^\varphi, \quad \mathcal{P}_*(t) = P_{\mathcal{X}_*(t)}^{\varphi_*}$$

are the sub-differentials of the potentials φ and φ_* where the Hamiltonian h attains its minimal values. The key inequality

$$\delta\mathcal{P}(t) \cdot \delta\mathcal{X}(t) \leq 2\varepsilon + 2e(\delta\mathcal{X}(t))$$

is almost the inequality (6). Like there we have

$$\varphi_*(\mathcal{X}) - \varphi_*(\mathcal{X}_*) \leq \mathcal{P}_* \cdot (\mathcal{X} - \mathcal{X}_*) + e(\mathcal{X} - \mathcal{X}_*),$$

$$\varphi(\mathcal{X}_*) - \varphi(\mathcal{X}) \leq \mathcal{P} \cdot (\mathcal{X}_* - \mathcal{X}) + e(\mathcal{X} - \mathcal{X}_*).$$

Adding these inequality and taking into account that

$$-2\varepsilon < \varphi_*(\mathcal{X}) - \varphi_*(\mathcal{X}_*) + \varphi(\mathcal{X}_*) - \varphi(\mathcal{X})$$

according to (9), we get our key inequality.

Our further proof is coordinate. Let $X = (X_1, X_2) \in \mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, $X_1 \in \mathbb{R}^{m_1}$, $X_2 \in \mathbb{R}^{m_2}$ be affine coordinates such that in the dual coordinates $(P_1, P_2) \in \mathcal{T}^*$ the Hamiltonian has a canonical form:

$$h(P_1, P_2) = P \cdot Q + |P_2|^2/2, \quad Q \in \mathcal{T}.$$

Then our trajectories satisfy the differential equations

$$\mathcal{X}^+(t) = Q + (0, \mathcal{P}_2(t)), \quad \mathcal{X}_*^+(t) = Q + (0, \mathcal{P}_{*2}(t))$$

because $h'_P = Q \cdot dP + P_2 \cdot dP_2$. Hence,

$$\delta\mathcal{X}_1^+(t) = 0, \quad \delta\mathcal{X}_2^+(t) = \delta\mathcal{P}_2(t).$$

Let $|X_1 - X_{*1}| < \alpha_1$, $|X_2 - X_{*2}| < \alpha_2$,

$$R(t) = |\delta\mathcal{X}_2(t)|^2, \quad 2e(\delta\mathcal{X}) \leq C|\delta\mathcal{X}|^2$$

for some $C \geq 0$. Then

$$|\delta\mathcal{X}_1(t)| = |X_1 - X_{*1}| < \alpha_1$$

and the above key inequality gives that

$$\delta\mathcal{P}(t) \cdot \delta\mathcal{X}(t) \leq 2\varepsilon + C|\delta\mathcal{X}(t)|^2.$$

Hence,

$$\begin{aligned} R^+(t) &= 2\delta\mathcal{P}_2(t) \cdot \delta\mathcal{X}_2(t) \leq \\ &\leq 4\varepsilon + 2C|\delta\mathcal{X}(t)|^2 - 2\delta\mathcal{P}_1(t) \cdot \delta\mathcal{X}_1(t) \leq \\ &\leq 4\varepsilon + 2C\alpha_1^2 + 2CR(t) + 4B\alpha_1. \end{aligned}$$

Solving the differential inequality and taking into account that $R(0) < \alpha_2^2$, we get:

$$R(t) < (2\varepsilon + C\alpha_1^2 + 2B\alpha_1) \frac{e^{2Ct} - 1}{C} + \alpha_2^2 e^{2Ct}.$$

Therefore, as $(\varepsilon, \alpha_1, \alpha_2) \rightarrow +0$

$$|\delta\mathcal{X}_1(t)| \rightarrow 0, \quad R(t) \rightarrow 0$$

if $t \leq t_*$. This proves the continuity when time is fixed: $t = t_*$. But if $t \leq t_*$ we can use the estimate

$$|\mathcal{X}_*(t) - \mathcal{X}_*(t_*)| \leq (|Q| + B)|t - t_*|,$$

and in the case $t \geq t_*$ -

$$|\mathcal{X}(t) - \mathcal{X}(t_*)| \leq (|Q| + B)|t - t_*|.$$

□

Now, in order to prove Theorem 5, it is enough to show that $D(\mathbf{g}) = \Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m$. Of course, we can uniquely define a continuous mapping

$$\mathbf{g} : \Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad \mathbf{g}|_{D(\mathbf{g})} = \mathbf{g}.$$

Indeed, $D(\mathbf{g})$ is dense in $\Phi_e \times \mathbb{R}^+ \times \mathbb{R}^m$ because all smooth potentials form a dense subset in Φ_e - that is shown with the help of the standard smoothing.

It turns out that any trajectory

$$\mathfrak{X}(t) = \mathbf{g}_\varphi^t(X_0)$$

is a solution of our Cauchy problem and, therefore, $\mathbf{g} = \mathbf{g}$. In order to show that we have to check

$$\mathfrak{X}^+(0) = X_0 \quad \text{and} \quad \mathfrak{X}^+(t) = \nabla_h(\varphi)(\mathfrak{X}(t)).$$

Of course, the first equality follows from the obvious fact $\mathbf{g}_\varphi^0 = \text{id}$ which is implied by $\mathbf{g}_\varphi^0 = \text{id}$. Why is the differential equation satisfied?

The point is that the class of our Cauchy problems is invariant with respect to adding constants to potentials, translations of X , positive shifts of t , and simultaneous dilations of the graphs of potentials, t , and X . Hence, the mappings \mathbf{g} and \mathbf{g} are invariant with respect to these transformations as well because the latter are continuous in the spaces Φ_e , \mathbb{R}^+ , and \mathbb{R}^m . In the terms of the mappings $\mathbf{g}_\varphi^t : \mathbb{R}^m \rightarrow \mathbb{R}^m$ it means the following.

1. Adding constants to potentials: $\mathfrak{g}_{\varphi+\text{const}}^t = \mathfrak{g}_{\varphi}^t$.
2. Translations of X : $\mathfrak{g}_{\varphi(X-X_0)}^t = \mathfrak{g}_{\varphi(X)}^t + X_0$.
3. Positive shifts of t : $\mathfrak{g}_{\varphi}^{t_1+t_2} = \mathfrak{g}_{\varphi}^{t_1} \circ \mathfrak{g}_{\varphi}^{t_2}$.
4. Simultaneous dilations of the graphs of potentials, t , and X :

$$\mathfrak{g}_{\lambda\varphi(X/\lambda)}^t(\lambda X_0) = \lambda \mathfrak{g}_{\varphi(X)}^t(X_0), \quad \lambda \geq 1.$$

(The last inequality guarantees that $\lambda\varphi(X/\lambda) \in \Phi_e$ if $\varphi \in \Phi_e$.)

The invariance property 3 implies that the differential equation is enough to be checked only for $t = 0$. According to the invariance properties 1 and 2, we can suppose that $\varphi(0) = 0$ and $X_0 = 0$. Besides, the invariance property 4 implies that

$$\mathfrak{g}_{\lambda\varphi(X/\lambda)}^t(0) = \lambda \mathfrak{X}(t/\lambda), \quad \lambda \geq 1.$$

Hence,

$$\mathfrak{X}^+(0) = \lim_{\lambda \rightarrow +\infty} \lambda \mathfrak{X}(1/\lambda) = \mathfrak{g}_{\varphi'}^1(0)$$

because $\mathfrak{X}(0) = X_0 = 0$ and

$$\lambda\varphi(X/\lambda) \rightarrow \varphi'(X) \text{ in } \Phi_e \text{ as } \lambda \rightarrow +\infty$$

in consequence of $\varphi(0) = 0$ and Theorems 3 and 4. But

$$\mathfrak{g}_{\varphi'}^1 = \nabla_h(\varphi)(0)$$

according to Lemma 3 that completes proving Theorem 5.

9 Computations for Figures

Figure 3 shows how particles move around nodes and end points of the shock. Figures 4–7 demonstrate what can generically happen with a cluster moving in the shock. In order to get all of these Figures except the one containing an end point (on the right of Figure 3), we use the following procedure for $k = 3$ or 4.

Let in a neighborhood of some point (\mathbf{x}_*, t_*) the limit potential be presented as the minimum of k smooth solutions of the Hamilton–Jacobi equation:

$$\varphi(\mathbf{x}, t) = \min \{ \varphi^1(\mathbf{x}, t), \dots, \varphi^k(\mathbf{x}, t) \}$$

where $\varphi(\mathbf{x}_*, t_*) = \varphi^1(\mathbf{x}_*, t_*) = \dots = \varphi^k(\mathbf{x}_*, t_*)$ and

$$\varphi_t^i(\mathbf{x}, t) + |\nabla \varphi^i(\mathbf{x}, t)|^2/2 + U(\mathbf{x}, t) = 0$$

for all $i = 1, \dots, k$. Therefore, the derivative $\varphi'_* = \varphi'_{\mathbf{x}_*, t_*}$ of the limit potential at the point (\mathbf{x}_*, t_*) is presented by the formula

$$\varphi'_*(\mathbf{q}, \tau) = \min_{i=1, \dots, k} \{ \mathbf{p}_i \cdot \mathbf{q} - \tau |\mathbf{p}_i|^2/2 \} - U_* \tau$$

where $\mathbf{p}_i = \nabla \varphi^i(\mathbf{x}_*, t_*)$ and $U_* = U(\mathbf{x}_*, t_*)$. Let us linearize the shock and the motion of particles around the point (\mathbf{x}_*, t_*) considering the derivative φ'_* instead of the limit potential itself. The shock of the derivative and the velocities of the particles are described with the help of the last displayed formula and Theorem 2 in the following way.

Let any three of the momenta $\mathbf{p}_1, \dots, \mathbf{p}_k$ do not belong to the same line and any four of them do not belong to the same circle.

For $\tau < 0$. If the minimum of the values $|\mathbf{p}_i \tau - \mathbf{q}|$, $i = 1, \dots, k$ is attained only for one index i_1 then the particle \mathbf{q} is outside of the shock and its velocity is \mathbf{p}_{i_1} . If this minimum is attained for two indices i_1, i_2 then the particle \mathbf{q} is being at a smooth point of the shock and its velocity is the midpoint of the segment $[\mathbf{p}_{i_1}, \mathbf{p}_{i_2}]$. If the minimum is attained for three indices i_1, i_2, i_3 then the particle \mathbf{q} is being situated at a node of the shock and its velocity is the center of the minimal disk containing the momenta $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \mathbf{p}_{i_3}$.

For $\tau > 0$. If the maximum of the values $|\mathbf{p}_i \tau - \mathbf{q}|$, $i = 1, \dots, k$ is attained only for one index i_1 then the particle \mathbf{q} is outside of the shock and its velocity is \mathbf{p}_{i_1} . If this maximum attained for two indices i_1, i_2 then the particle \mathbf{q} is being at a smooth point of the shock and its velocity is the midpoint of the segment $[\mathbf{p}_{i_1}, \mathbf{p}_{i_2}]$. If the maximum is attained for three indices i_1, i_2, i_3 then the particle \mathbf{q} is being situated at a node of the shock and its velocity is the center of the minimal disk containing the momenta $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \mathbf{p}_{i_3}$. Besides, the center of the minimal disk containing all of the points $\mathbf{p}_1 \tau, \dots, \mathbf{p}_k \tau$ belongs to the shock and the unique trajectory coming from the origin $(\mathbf{q}, \tau) = 0$.

REMARK. If $\tau > 0$, the shock of the derivative is the Voronoi diagram of the points $\mathbf{p}_1 \tau, \dots, \mathbf{p}_k \tau$. If $\tau < 0$, the shock is the set being analogous to the Voronoi diagram of the points $\mathbf{p}_1 \tau, \dots, \mathbf{p}_k \tau$ but defined by multiple maxima of the distance (not minima).

Let $k = 3$ and the momenta $\mathbf{p}_1, \mathbf{p}_2$, and \mathbf{p}_3 do not belong to the same straight line. Then the shock of the derivative φ'_* has a node at the center of the circle containing the points $\tau \mathbf{p}_1, \tau \mathbf{p}_2$, and $\tau \mathbf{p}_3$ and the velocity of this node is the center of the circle passing through the momenta $\mathbf{p}_1, \mathbf{p}_2$, and \mathbf{p}_3 . But the velocity of the particle situated at the node at a given time is the center of the minimal disk containing the momenta $\mathbf{p}_1, \mathbf{p}_2$, and \mathbf{p}_3 . Of course, these centers do not always coincide.

Namely, there are two generic possibilities: the triangle with the vertices $\mathbf{p}_1, \mathbf{p}_2$, and \mathbf{p}_3 can be obtuse or acute. If the triangle is obtuse then the velocities of the node and the particle are different and

the particle leaves the node. Otherwise, if the triangle is acute then the velocities coincide and the particle stays at the node. It is convenient to apply the above procedure in a frame of reference connected with the node. In this case the shock does not change with time, $|\mathbf{p}_1|^2 = |\mathbf{p}_2|^2 = |\mathbf{p}_3|^2$, and we get the shock and velocities shown in Figure 3 on the left (the triangle is obtuse) and in the middle (the triangle is acute).

REMARK. There is another simple explanation of the difference between these possibilities. Namely, in a frame of reference connected with the node the derivative φ'_* has no extremum on the left and does have a maximum in the middle of the Figure 3.

At separate times our triangle can become right – it is shown in Figures 4 and 5 on the left.

Let $k = 4$, and $\mathbf{p}_1, \dots, \mathbf{p}_4$ is a *generic* configuration of momenta. It means that they do not belong to the same circle and any three of them do not belong to the same line and do not form a right triangle. Such generic configurations have many connected components, the components with four obtuse triangles are called *totally obtuse*. A component is called *narrow* if the boundary of the minimal disk containing all the momenta passes only through two of them. A component is called *wide* if the boundary of the minimal disk containing all the momenta passes through three of them. Of course, any totally obtuse configuration is narrow. Each component, except the totally obtuse ones, defines one of the types of behavior of clusters during the fifth or sixth transitions from Figure 2 – all these types are shown in the middle and on the right of Figures 4–7. Namely, taking a configuration from a connected component we apply the above procedure which shows the following.

Each triangle formed by three of the momenta defines a node before or after the transition. If this triangle is acute then there is a growing cluster at the node. If the configuration is totally obtuse then there are no clusters at all and we ignore it. If the convex hull of the momenta is a triangle then the fifth transition occurs, if the convex hull is a quadrangle – the sixth one does. If the configuration is not totally obtuse then there is a cluster after the transition – its trajectory comes from the origin. If the configuration is narrow then this cluster is stable, if it is wide then the cluster is growing.

It remains to compute the velocities around an end point shown on the right of Figure 3. Let

$$\varphi(\mathbf{x}, t) = \min_{\xi} \{F(\xi, \mathbf{x}, t)\}$$

where F is a family of smooth solutions of the

Hamilton–Jacobi equation:

$$F_t(\xi, \mathbf{x}, t) + |\nabla_{\mathbf{x}} F(\xi, \mathbf{x}, t)|^2/2 + U(\mathbf{x}, t) = 0$$

such that the function $F(\cdot, \mathbf{x}_*, t_*)$ has the simplest minimum from the degenerate ones at the point $\xi = 0$. (In Singularity Theory this minimum is called A_3 .) It means:

$$\begin{aligned} F(\xi, \mathbf{x}, t) - F(0, \mathbf{x}, t) = \\ = Aa^4 + 2 \sum_{i=1}^k B_i a^2 b_i + \sum_{i,j=1}^k C_{ij} b_i b_j + \\ + \alpha(\mathbf{q}, \tau) a + \beta(\mathbf{q}, \tau) a^2 + \sum_{i=1}^k \gamma_i(\mathbf{q}, \tau) b_i + \dots, \\ \xi = (a, b_1, \dots, b_k), \quad \mathbf{q} = \mathbf{x} - \mathbf{x}_*, \quad \tau = t - t_*; \end{aligned}$$

where the first part of the right side is a positive definite quadratic form of a^2, b_1, \dots, b_k ; $\alpha, \beta, \gamma_1, \dots, \gamma_k$ are linear forms of (\mathbf{q}, τ) ; and the dots denote the high order terms. This minimum representation gives us the following.

1. In a neighborhood of the point (\mathbf{x}_*, t_*) the shock of the limit potential φ in the space-time is approximated by the semi-hyperplane

$$\begin{aligned} \alpha(\mathbf{q}, \tau) = 0, \\ \det \begin{pmatrix} \beta(\mathbf{q}, \tau) & \gamma_1(\mathbf{q}, \tau) & \dots & \gamma_k(\mathbf{q}, \tau) \\ B_1 & C_{11} & \dots & C_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ B_k & C_{k1} & \dots & C_{kk} \end{pmatrix} \leq 0. \end{aligned}$$

2. If $\mathbf{p}_* = \nabla_{\mathbf{x}} F(0, \mathbf{x}_*, t_*)$ then

$$\varphi'_{\mathbf{x}_*, t_*}(\mathbf{q}, \tau) = \mathbf{p}_* \cdot \mathbf{q} - \tau |\mathbf{p}_*|^2/2 - U(\mathbf{x}_*, t_*) \tau.$$

Hence, Theorem 2 implies that $\mathbf{u}(\mathbf{x}_*, t_*) = \mathbf{p}_*$ and the space-time vector $(\mathbf{p}_*, 1)$ is tangent to the trajectory starting at the point (\mathbf{x}_*, t_*) .

3. Differentiating the Hamilton–Jacobi equation we get:

$$F_{at} + \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{x}} F_a = 0, \quad F_{b_i t} + \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{x}} F_{b_i} = 0,$$

$$F_{aat} + \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{x}} F_{aa} = -\nabla_{\mathbf{x}} F_a \cdot \nabla_{\mathbf{x}} F_a.$$

After substituting $\xi = 0$, $\mathbf{q} = 0$, and $\tau = 0$ these equalities give:

$$\alpha(\mathbf{p}_*, 1) = 0, \quad \gamma_i(\mathbf{p}_*, 1) = 0, \quad \beta(\mathbf{p}_*, 1) < 0.$$

Therefore, the trajectory starting at the end point (\mathbf{x}_*, t_*) of the shock goes inside it because its tangent vector $(\mathbf{p}_*, 1)$ belongs to the semi-hyperplane from the first observation. In order to see that, we have to take into account the third observation and that

$$\det \begin{pmatrix} C_{11} & \dots & C_{1k} \\ \vdots & \ddots & \vdots \\ C_{k1} & \dots & C_{kk} \end{pmatrix} > 0$$

according to the positive definiteness.

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